

14.5 | Oscillations in an LC Circuit

Learning Objectives

By the end of this section, you will be able to:

- Explain why charge or current oscillates between a capacitor and inductor, respectively, when wired in series
- Describe the relationship between the charge and current oscillating between a capacitor and inductor wired in series

It is worth noting that both capacitors and inductors store energy, in their electric and magnetic fields, respectively. A circuit containing both an inductor (L) and a capacitor (C) can oscillate without a source of emf by shifting the energy stored in the circuit between the electric and magnetic fields. Thus, the concepts we develop in this section are directly applicable to the exchange of energy between the electric and magnetic fields in electromagnetic waves, or light. We start with an idealized circuit of zero resistance that contains an inductor and a capacitor, an **LC circuit**.

An LC circuit is shown in **Figure 14.16**. If the capacitor contains a charge q_0 before the switch is closed, then all the energy of the circuit is initially stored in the electric field of the capacitor (**Figure 14.16(a)**). This energy is

$$U_C = \frac{1}{2} \frac{q_0^2}{C}. \quad (14.33)$$

When the switch is closed, the capacitor begins to discharge, producing a current in the circuit. The current, in turn, creates a magnetic field in the inductor. The net effect of this process is a transfer of energy from the capacitor, with its diminishing electric field, to the inductor, with its increasing magnetic field.

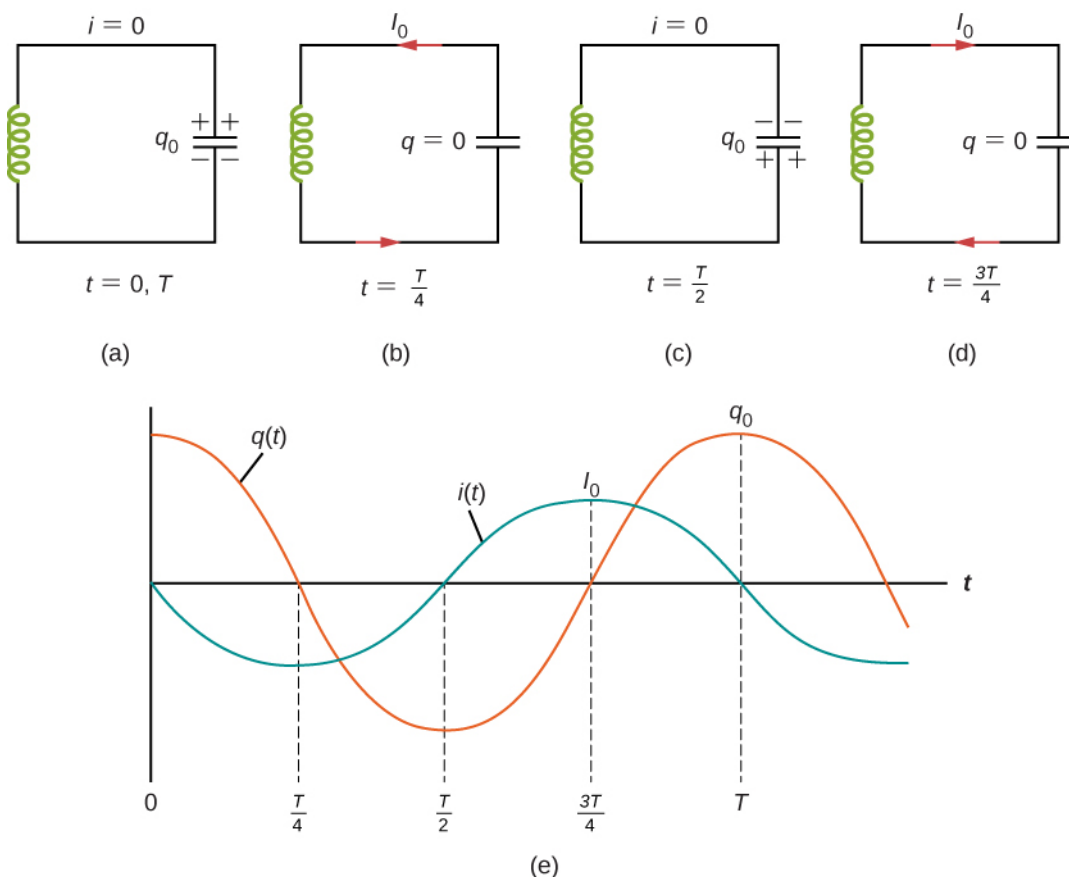


Figure 14.16 (a–d) The oscillation of charge storage with changing directions of current in an LC circuit. (e) The graphs show the distribution of charge and current between the capacitor and inductor.

In **Figure 14.16(b)**, the capacitor is completely discharged and all the energy is stored in the magnetic field of the inductor. At this instant, the current is at its maximum value I_0 and the energy in the inductor is

$$U_L = \frac{1}{2}LI_0^2. \quad (14.34)$$

Since there is no resistance in the circuit, no energy is lost through Joule heating; thus, the maximum energy stored in the capacitor is equal to the maximum energy stored at a later time in the inductor:

$$\frac{1}{2} \frac{q_0^2}{C} = \frac{1}{2}LI_0^2. \quad (14.35)$$

At an arbitrary time when the capacitor charge is $q(t)$ and the current is $i(t)$, the total energy U in the circuit is given by

$$\frac{q^2(t)}{2C} + \frac{Li^2(t)}{2}.$$

Because there is no energy dissipation,

$$U = \frac{1}{2} \frac{q^2}{C} + \frac{1}{2}Li^2 = \frac{1}{2} \frac{q_0^2}{C} = \frac{1}{2}LI_0^2. \quad (14.36)$$

After reaching its maximum I_0 , the current $i(t)$ continues to transport charge between the capacitor plates, thereby recharging the capacitor. Since the inductor resists a change in current, current continues to flow, even though the capacitor is discharged. This continued current causes the capacitor to charge with opposite polarity. The electric field of the capacitor increases while the magnetic field of the inductor diminishes, and the overall effect is a transfer of energy from the inductor back to the capacitor. From the law of energy conservation, the maximum charge that the capacitor re-acquires is q_0 .

However, as **Figure 14.16(c)** shows, the capacitor plates are charged *opposite* to what they were initially.

When fully charged, the capacitor once again transfers its energy to the inductor until it is again completely discharged, as shown in **Figure 14.16(d)**. Then, in the last part of this cyclic process, energy flows back to the capacitor, and the initial state of the circuit is restored.

We have followed the circuit through one complete cycle. Its electromagnetic oscillations are analogous to the mechanical oscillations of a mass at the end of a spring. In this latter case, energy is transferred back and forth between the mass, which has kinetic energy $mv^2/2$, and the spring, which has potential energy $kx^2/2$. With the absence of friction in the mass-spring system, the oscillations would continue indefinitely. Similarly, the oscillations of an LC circuit with no resistance would continue forever if undisturbed; however, this ideal zero-resistance LC circuit is not practical, and any LC circuit will have at least a small resistance, which will radiate and lose energy over time.

The frequency of the oscillations in a resistance-free LC circuit may be found by analogy with the mass-spring system. For the circuit, $i(t) = dq(t)/dt$, the total electromagnetic energy U is

$$U = \frac{1}{2}Li^2 + \frac{1}{2} \frac{q^2}{C}. \quad (14.37)$$

For the mass-spring system, $v(t) = dx(t)/dt$, the total mechanical energy E is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \quad (14.38)$$

The equivalence of the two systems is clear. To go from the mechanical to the electromagnetic system, we simply replace m by L , v by i , k by $1/C$, and x by q . Now $x(t)$ is given by

$$x(t) = A \cos(\omega t + \phi) \quad (14.39)$$

where $\omega = \sqrt{k/m}$. Hence, the charge on the capacitor in an LC circuit is given by

$$q(t) = q_0 \cos(\omega t + \phi) \quad (14.40)$$

where the angular frequency of the oscillations in the circuit is

$$\omega = \sqrt{\frac{1}{LC}}. \quad (14.41)$$

Finally, the current in the LC circuit is found by taking the time derivative of $q(t)$:

$$i(t) = \frac{dq(t)}{dt} = -\omega q_0 \sin(\omega t + \phi). \quad (14.42)$$

The time variations of q and I are shown in **Figure 14.16(e)** for $\phi = 0$.

Example 14.6

An LC Circuit

In an LC circuit, the self-inductance is 2.0×10^{-2} H and the capacitance is 8.0×10^{-6} F. At $t = 0$, all of the energy is stored in the capacitor, which has charge 1.2×10^{-5} C. (a) What is the angular frequency of the oscillations in the circuit? (b) What is the maximum current flowing through circuit? (c) How long does it take the capacitor to become completely discharged? (d) Find an equation that represents $q(t)$.

Strategy

The angular frequency of the LC circuit is given by **Equation 14.41**. To find the maximum current, the maximum energy in the capacitor is set equal to the maximum energy in the inductor. The time for the capacitor to become discharged if it is initially charged is a quarter of the period of the cycle, so if we calculate the period of the oscillation, we can find out what a quarter of that is to find this time. Lastly, knowing the initial charge and angular frequency, we can set up a cosine equation to find $q(t)$.

Solution

- a. From **Equation 14.41**, the angular frequency of the oscillations is

$$\omega = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(2.0 \times 10^{-2} \text{ H})(8.0 \times 10^{-6} \text{ F})}} = 2.5 \times 10^3 \text{ rad/s}.$$

- b. The current is at its maximum I_0 when all the energy is stored in the inductor. From the law of energy conservation,

$$\frac{1}{2}LI_0^2 = \frac{1}{2}\frac{q_0^2}{C},$$

so

$$I_0 = \sqrt{\frac{1}{LC}}q_0 = (2.5 \times 10^3 \text{ rad/s})(1.2 \times 10^{-5} \text{ C}) = 3.0 \times 10^{-2} \text{ A}.$$

This result can also be found by an analogy to simple harmonic motion, where current and charge are the velocity and position of an oscillator.

- c. The capacitor becomes completely discharged in one-fourth of a cycle, or during a time $T/4$, where T is the period of the oscillations. Since

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.5 \times 10^3 \text{ rad/s}} = 2.5 \times 10^{-3} \text{ s},$$

the time taken for the capacitor to become fully discharged is $(2.5 \times 10^{-3} \text{ s})/4 = 6.3 \times 10^{-4} \text{ s}$.

- d. The capacitor is completely charged at $t = 0$, so $q(0) = q_0$. Using **Equation 14.20**, we obtain

$$q(0) = q_0 = q_0 \cos \phi.$$

Thus, $\phi = 0$, and

$$q(t) = (1.2 \times 10^{-5} \text{ C}) \cos(2.5 \times 10^3 t).$$

Significance

The energy relationship set up in part (b) is not the only way we can equate energies. At most times, some energy is stored in the capacitor and some energy is stored in the inductor. We can put both terms on each side of the equation. By examining the circuit only when there is no charge on the capacitor or no current in the inductor, we simplify the energy equation.



14.10 Check Your Understanding The angular frequency of the oscillations in an LC circuit is 2.0×10^3 rad/s. (a) If $L = 0.10 \text{ H}$, what is C ? (b) Suppose that at $t = 0$, all the energy is stored in the inductor. What is the value of ϕ ? (c) A second identical capacitor is connected in parallel with the original capacitor. What is the angular frequency of this circuit?

14.6 | RLC Series Circuits

Learning Objectives

By the end of this section, you will be able to:

- Determine the angular frequency of oscillation for a resistor, inductor, capacitor (RLC) series circuit
- Relate the RLC circuit to a damped spring oscillation

When the switch is closed in the **RLC circuit** of **Figure 14.17(a)**, the capacitor begins to discharge and electromagnetic energy is dissipated by the resistor at a rate $i^2 R$. With U given by **Equation 14.19**, we have

$$\frac{dU}{dt} = \frac{q}{C} \frac{dq}{dt} + Li \frac{di}{dt} = -i^2 R \quad (14.43)$$

where i and q are time-dependent functions. This reduces to

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = 0. \quad (14.44)$$